Midterm Exam Calculus 2

18 March 2019, 9:00-11:00

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[8+7+5 Points.]**
   Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
   $$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} .$$
   (a) Use the definition of partial derivatives to calculate $f_x(0, 0)$ and $f_y(0, 0)$.
   (b) Let $a \in \mathbb{R}$ with $a \neq 0$, and let $\mathbf{r}(t) = (t, at)$. Show that the composite function $f \circ \mathbf{r} : \mathbb{R} \to \mathbb{R}$ is differentiable at $t = 0$.
   (c) Compute $\nabla f(0, 0) \cdot \mathbf{r}'(0)$. Reconcile this result with your result in part (b) to conclude on the differentiability of f at $(x, y) = (0, 0)$.

2. **[10+5+10 Points.]**
   Consider the curve parametrized by $\mathbf{r} : [0, 1] \to \mathbb{R}^3$ with
   $$\mathbf{r}(t) = (t \cos t - \sin t)\, \mathbf{i} + (t \sin t + \cos t)\, \mathbf{j} + \mathbf{k}.$$
   (a) Determine the parametrization by arc length.
   (b) For each point on the curve, determine a unit tangent vector.
   (c) At each point on the curve, determine the curvature of the curve.

3. **[10+10+5 Points.]**
   (a) Use the method of Lagrange multipliers to find the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on the unit sphere $x^2 + y^2 + z^2 = 1$ where $f(x, y, z) = x + y - z$ assumes its maximum value and its minimum value, respectively.
   (b) Show that the tangent plane of the unit sphere at the point (x_1, y_1, z_1) is given by the equation $f(x, y, z) = f(x_1, y_1, z_1)$ and the tangent plane of the unit sphere at the point (x_2, y_2, z_2) is given by the equation $f(x, y, z) = f(x_2, y_2, z_2)$.
   (c) Let $(x_0, y_0, z_0) \in \mathbb{R}^3$. Show that f agrees with its linearization at (x_0, y_0, z_0).

4. **[20 Points.]**
   Determine
   $$\iiint_W (2 + \sqrt{x^2 + y^2})\, \mathrm{d}V,$$
   where $W = \{(x, y, z) \in \mathbb{R}^3 \,|\, 2(x^2 + y^2)^{1/2} \leq z \leq 1\}$.

Solutions

1. (a) Following the definition, the partial derivative of f with respect to x at $(x,y) = (0,0)$ is

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^2\cdot}{h^2+0^2}-0}{h} = \lim_{h\to 0} 0 = 0.$$

Similarly

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0^2\cdot h}{0^2+h^2}-0}{h} = \lim_{h\to 0} 0 = 0.$$

(b) Let $g = f\circ \mathbf{r}$. Then

$$g(t) = \begin{cases} \frac{at^3}{t^2+a^2t^2} & \text{if } t\neq 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For the differentiability of g at $t=0$ consider for $h\neq 0$, the difference quotient

$$\frac{g(h)-g(0)}{h} = \frac{\frac{ah^3}{h^2+a^2h^2}-0}{h} = \frac{ah^3}{h^3+a^2h^3} = \frac{a}{1+a^2}.$$

As the difference quotient has a limit for $h\to 0$ we conclude that g is differentiable at $t=0$ and the derivative is $g'(0) = \frac{a}{1+a^2}$.

(c) From part (a) we have $\nabla f(0,0) = (0,0)$. We have $\mathbf{r}'(0) = (1,a)$. So $\nabla f(0,0)\cdot \mathbf{r}'(0) = 0$. If f would be differentiable at $(x,y) = (0,0)$ then by the Chain Rule the derivative of $f\circ \mathbf{r}$ at $t=0$ would be $\nabla f(0,0)\cdot\mathbf{r}'(0) = 0$ which does not agree with the result in part (b). We conclude that f is not differentiable at $(x,y) = (0,0)$.

2. (a) The tangent vector

$$\mathbf{r}'(t) = (\cos t - t\sin t - \cos t)\,\mathbf{i} + (\sin t + t\cos t - \sin t)\,\mathbf{j} + 0\,\mathbf{k} = -t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j}$$

has length

$$\|\mathbf{r}'(t)\|| = \left((-t\sin t)^2 + (t\cos t)^2\right)^{1/2} = \left(t^2\right)^{1/2} = |t| = t\,,$$

where we used that $t\in[0,1]$ and hence t is positive in the last equality. The arc length is hence

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau = \int_0^t \tau\,\mathrm{d}\tau = \frac{1}{2}t^2.$$

Note that $s(0) = 0$ and $s(1) = \frac{1}{2}$ where the latter is the length of the curve. Inverting for t gives

$$t(s) = \sqrt{2s}.$$

The parametrization by arc length is hence given by

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = (\sqrt{2s}\cos\sqrt{2s} - \sin\sqrt{2s})\,\mathbf{i} + (\sqrt{2s}\sin\sqrt{2s} + \cos\sqrt{2s})\,\mathbf{j} + \,\mathbf{k}$$

with $s\in[0,\frac{1}{2}]$.

(b) The unit tangent vector at the point $\mathbf{r}(t)$, $t \in [0,1]$, is given by

$$\mathbf{T} = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \frac{1}{t}\left(-t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j}\right) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

which agrees with

$$\mathbf{T} = \frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}s}(s)$$

for $s = \frac{1}{2}t^2$.

(c) Viewing the unit tangent vector in part (b) to be given as a function of s the curvature is given by

$$\begin{aligned}\kappa &= \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}}{\mathrm{d}s}\left(-\sin\sqrt{2s}\,\mathbf{j} + \cos\sqrt{2s}\,\mathbf{k}\right)\right\| \\ &= \left\|-\frac{1}{\sqrt{2s}}\sin\sqrt{2s}\,\mathbf{j} + \frac{1}{\sqrt{2s}}\cos\sqrt{2s}\,\mathbf{k}\right\| \\ &= \frac{1}{\sqrt{2s}}\end{aligned}$$

which for $t = \sqrt{2s}$, agrees with

$$\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right\| \frac{1}{\left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right\|}$$

when viewing the unit tangent vector in part (b) as a function of t.

3. (a) Let $g(x,y,z) = x^2+y^2+z^2$. Then the unit sphere is the level set of g with value 1. At an extremum of f under the constraint $g(x,y,z) = 1$ there is according to the theorem on Lagrange multipliers a $\lambda \in \mathbb{R}$ such that $\lambda\nabla f(x,y,z) = \nabla g(x,y,z)$. Together with the constraint $g(x,y,z) = 1$ this gives the following four scalar equations:

$$\begin{aligned}\lambda f_x(x,y,z) &= g_x(x,y,z), \\ \lambda f_y(x,y,z) &= g_y(x,y,z), \\ \lambda f_z(x,y,z) &= g_z(x,y,z), \\ x^2+y^2+z^2 &= 1\end{aligned}$$

i.e.

$$\begin{aligned}\lambda &= 2x, \\ \lambda &= 2y, \\ -\lambda &= 2z, \\ x^2+y^2+z^2 &= 1.\end{aligned}$$

We see that $x = y = -z$ which needs to be satisfied together with $x^2+y^2+z^2 = 1$ (λ is then given by, e.g., $2x$). This leads to the two points

$$(x_1,y_1,z_1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

and

$$(x_2,y_2,z_2) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

From the Weierstrass Extreme Value Theorem we know that f assumes its maximum and minimum values on the unit sphere. From $f(x_1,y_1,z_1) = \sqrt{3}$ and $f(x_2,y_2,z_2) = -\sqrt{3}$ we see that (x_1,y_1,z_1) is the point where f assumes its maximum and (x_2,y_2,z_2) is the point where f assumes its minimum.

(b) The tangent plane of the unit sphere at (x_k, y_k, z_k) is orthogonal to $\nabla g(x_k, y_k, z_k)$ for $k = 1, 2$. The tangent plane at (x_k, y_k, z_k) is hence given by $\nabla g(x_k, y_k, z_k) \cdot (x - x_k, y - y_k, z - z_k) = 0$. For (x_1, y_1, z_1) this gives

$$
\begin{aligned}
& 2\Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\Big) \cdot \Big(x - \frac{1}{\sqrt{3}}, y - \frac{1}{\sqrt{3}}, z + \frac{1}{\sqrt{3}}\Big) = 0 \\
\Leftrightarrow\quad & \frac{1}{\sqrt{3}}x - \frac{1}{3} + \frac{1}{\sqrt{3}}y - \frac{1}{3} - \frac{1}{\sqrt{3}}z - \frac{1}{3} = 0 \\
\Leftrightarrow\quad & x + y - z = \sqrt{3}.
\end{aligned}
$$

As $\sqrt{3} = f(x_1, y_1, z_1)$ we see that the tangent plane of the unit sphere at (x_1, y_1, z_1) satisfies $f(x, y, z) = f(x_1, y_1, z_1)$.
Similarly for (x_x, y_x, z_x) then tangent plane is given by

$$
\begin{aligned}
& 2\Big(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big) \cdot \Big(x + \frac{1}{\sqrt{3}}, y + \frac{1}{\sqrt{3}}, z - \frac{1}{\sqrt{3}}\Big) = 0 \\
\Leftrightarrow\quad & -\frac{1}{\sqrt{3}}x - \frac{1}{3} - \frac{1}{\sqrt{3}}y - \frac{1}{3} + \frac{1}{\sqrt{3}}z - \frac{1}{3} = 0 \\
\Leftrightarrow\quad & x + y - z = -\sqrt{3}.
\end{aligned}
$$

As $-\sqrt{3} = f(x_2, y_2, z_2)$ we see that the tangent plane of the unit sphere at (x_2, y_2, z_2) satisfies $f(x, y, z) = f(x_2, y_2, z_2)$.

(c) The linearization of f at (x_0, y_0, z_0) is given by

$$
\begin{aligned}
L(x, y, z) &= f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \\
&= x_0 + y_0 - z_0 + 1 \cdot (x - x_0) + 1 \cdot (y - y_0) - 1 \cdot (z - z_0) \\
&= x + y - z
\end{aligned}
$$

which agrees with $f(x, y, z)$.

4. The cylinder geometry suggests to use cylinder coordinates, i.e. $x = r\cos\theta$, $y = r\sin\theta$ and z stays z. Then

$$
\begin{aligned}
\iiint_W (2 + \sqrt{x^2} + y^2)\,\mathrm{d}V &= \int_0^1 \int_1^{z/2} \int_0^{2\pi} (2 + r)\, r\mathrm{d}\theta\mathrm{d}r\mathrm{d}z \\
&= 2\pi \int_0^1 \int_1^{z/2} (2 + r) r \,\mathrm{d}r\mathrm{d}z \\
&= 2\pi \int_0^1 \left(r^2 + \frac{1}{3} r^3 \Big|_{r=0}^{r=z/2} \right) \mathrm{d}z \\
&= 2\pi \int_0^1 \left(\frac{1}{4} z^2 + \frac{1}{24} z^3 \right) \mathrm{d}z \\
&= 2\pi \left(\frac{1}{12} z^3 + \frac{1}{96} z^4 \Big|_{z=0}^{z=1} \right) \\
&= 2\pi \left(\frac{1}{12} + \frac{1}{96} \right) \\
&= \frac{9}{48}\pi\,.
\end{aligned}
$$